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FINITE DEFORMATION ELASTO-PLASTIC SOLUTION FOR THE PURE BENDING PROBLEM OF A WIDE PLATE OF ELASTIC LINEAR-HARDENING MATERIAL

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Abstract—An exact elasto-plastic analytical solution for a finitely deformed plane strain wide plate of elastic linear-hardening material subjected to pure bending is derived in this paper using a tensorial formulation. This solution is based on a finite-strain version of Hencky's deformation theory, the von Mises yield criterion and the incompressibility assumption. The Hencky (logarithmic) strain tensor is adopted to measure the finite deformations, with the undeformed flat plate as the reference configuration. The constitutive relations are derived in tensor form using the Hencky strain and the Cauchy stress. With both the linear–elastic and strain-hardening–plastic material responses included, the present solution can represent the whole bending process of the plate from its initial flat state to its final curved state with arbitrarily large deformation. It is shown that this solution exhibits general characteristics, from which three specific solutions of practical value can be obtained. Being expressed in both tensor and component forms, the present exact solution for the stress and strain fields furnishes a new and systematic analytical pattern for the elasto-plastic analysis and strength design of a strain-hardening wide plate subjected to large-deformation pure bending.

1. INTRODUCTION

Sheet metal forming by bending has wide applications in manufacturing industries. Almost all cylindrical pressure vessel shells are made from flat sheet metals (wide plates) by sequential bending using specially designed machines. Therefore, it is of great importance to have a better understanding of the stress and strain fields within the deformed plate during the whole bending process.

The simple case of a plane strain wide plate subjected to pure bending has long been an interesting research topic in this field. Since such a bending problem involves large deformations and the plate material exhibits linear–elastic, strain-hardening–plastic characteristics, an appropriate theoretical analysis for this kind of problem should be based on a finite deformation elasto-plastic model and should account for both the elastic and strain-hardening–plastic responses of the material.

However, owing to the mathematical difficulties arising from the coupled geometric and material non-linearities, such a systematic and complete analytical solution has, to the author's knowledge, not yet been reported. It seems that the principal theoretical solutions available for the plane strain pure bending problem, some of which were summarized in a recent monograph by Lubliner (1990), are all subject to some restrictions. For example, the solution obtained by Lubahn and Sachs (1950) was a fully plastic non-hardening solution and also a small-deformation solution with engineering strain adopted in the analysis; the solutions for the stress and displacement fields presented in Shaffer and House (1955, 1957), respectively, were based on an elastic–perfectly plastic material model and infinitesimal strain assumptions; in Denton (1966) the work-hardening behaviour of the material was included and the Prandtl–Reuss incremental theory was used to derive an elasto-plastic solution. However, this solution was only valid for small strains (up to 2%) and for very thin plates. Unlike these solutions, in Chen's (1962) solution logarithmic strains were applied to account for finite deformations of the plate and a power-law hardening model was used to represent the strain-hardening effect of the material, but the elastic response of the plate was excluded and also no strain–displacement analysis was carried out. In addition, as with the above-mentioned solutions, all analyses in Chen (1962) were relative to an unspecified curved plate which might be regarded as a reference configuration. As a result, this solution, like others, could not represent the deformation process of the plate from its initial flat state to this "undeformed" curved state. Besides these restrictions, it must also be noted that all these solutions were actually obtained by treating the pure bending problem as a statically determinate problem of plasticity theory, where the stress components were derived using only the equilibrium equations and chosen yield criterion, with no use being made of the constitutive equations. Consequently, the associated strain–displacement solutions, if any, should have been subjected to the compatibility equations for further checking.

The only elasto-plastic solution of the plane strain pure bending problem that is within the framework of finite deformation plasticity and with the unstressed flat plate as the reference configuration is due to Triantafyllidis (1980), which was concisely reported as a prebifurcation solution. This solution was based on an elastic power-law hardening material model [different from the pure power-law model used in Chen (1962)] and a finite-strain version of J_2 deformation theory. However, since this stress solution was expressed in terms of the curvature of the unstretched fibre layer with fixed length (i.e. the neutral surface) which remained unknown, it is not a complete solution even for that specific material model. Therefore, more satisfactory solutions that can deal with the whole finite deformation plate bending process and account for both the linear–elastic and strain-hardening–plastic material responses as well as be convenient to use in engineering design are still needed.

The present study attempts to fulfil this need. In Section 2, a finite deformation analysis is carried out by using the initially undeformed flat plate as the reference configuration. Incompressibility is assumed to facilitate the kinematic analysis, and the Hencky (logarithmic) strains are adopted to measure the finite deformations. This is followed by the development of the constitutive relations required to solve the deformation problem in Section 3. Here an elastic linear-hardening material model is applied, and a finite-strain version of Hencky's deformation theory is presented in tensor form. Following these preliminary considerations, the boundary-value problem is formulated in Section 4 and then solved in a systematic manner to obtain the elasto-plastic stress and strain fields. Both finite deformations and elastic–plastic material responses are considered in this formulation and, as a result, the solution can represent the complete elasto-plastic plate bending process, i.e. from its initial flat state to its final curved configuration with arbitrarily large curvature (but without bifurcation).

Section 5 is devoted to extensions and applications of the solution. Three specific solutions of practical value are presented as special cases of the general solution. Also, some numerical results are provided in this section and a few useful conclusions are drawn from these results as well as from the analytical expressions. In the last section, several conclusions are presented concerning the analytical developments described in the paper. Moreover, the limitations of the solution, which require further investigation, are indicated in this section.

2. KINEMATICS OF FINITE DEFORMATION

To derive the kinematic relations of the problem, we need to decide on a deformed configuration as well as a reference configuration with respect to which the deformation is measured. For the plane strain pure bending problem with large deformations but no strain hardening, it has been shown analytically by Hill (1950), on the basis of a displacement field analysis of a fully yielded wide plate, that during the bending process: (1) the cross-sectional planes remain plane, (2) a layer of radius r ($a \leqslant r \leqslant b$) turns into a cylindrical surface, and (3) the plate thickness remains unchanged. Based on this work, we assume that for the strain-hardening pure bending problem, which is physically similar to the former, the first two points are still valid while the last point is no longer effective. With these arguments, we can then use the configurations shown in Fig. 1 for our purpose, where the deformed configuration, Fig. 1(b), is part of a cylindrical shell (either thin- or thick-walled).

The kinematic analysis of such a problem may be found in most standard texts or treatises [see, for example, Ogden (1984)]. To emphasize the present differences and to facilitate our analysis, it is reformulated here.

We adopt a semi-inverse approach by taking $r = f(X)$, $\theta = g(Y)$, $z = Z$ for the plane strain problem at hand, where (X, Y, Z) and (r, θ, z) are the Lagrangian and Eulerian coordinates of a material point, respectively. The mapping $\mathbf{X} \to \mathbf{r}$, i.e.

$$\mathbf{X} = X\mathbf{e}_1 + Y\mathbf{e}_2 + Z\mathbf{e}_3, \quad -T \leqslant X \leqslant T, \quad -L \leqslant Y \leqslant L, \quad -W \leqslant Z \leqslant W \to$$
$$\mathbf{r} = r\mathbf{e}_r(\theta) + z\mathbf{k}, \quad r = f(X), \quad \theta = g(Y), \quad z = Z,$$
$$\mathbf{k} = \mathbf{e}_3; \quad a \leqslant r \leqslant b, \quad -\alpha \leqslant \theta \leqslant \alpha, \quad -W \leqslant z \leqslant W \tag{1}$$

represents the deformation specified in Fig. 1, where $2T$, $2L$, and $2W$ are the initial plate thickness, length, and width respectively; and a, b, and 2α are, respectively, the concave, convex radii, and the bending angle of the plate in the deformed configuration. The deformation gradient $\mathbf{F}$, defined by $\mathrm{d}\mathbf{r} = \mathbf{F}\,\mathrm{d}\mathbf{X}$, can then easily be shown to be

$$\mathbf{F} = f'(X)\mathbf{e}_r \otimes \mathbf{e}_1 + f(X)\,g'(Y)\mathbf{e}_\theta \otimes \mathbf{e}_2 + \mathbf{k} \otimes \mathbf{e}_3, \tag{2}$$

where the prime denotes differentiation with respect to the argument of the function. This may be rewritten as

$$\mathbf{F} = \mathbf{R}[f'(X)\mathbf{e}_1 \otimes \mathbf{e}_1 + f(X)\,g'(Y)\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3], \tag{3}$$

where

$$\mathbf{R} = \mathbf{e}_r \otimes \mathbf{e}_1 + \mathbf{e}_\theta \otimes \mathbf{e}_2 + \mathbf{k} \otimes \mathbf{e}_3 \tag{4}$$

is a (rigid-body) rotation (or proper orthogonal tensor), i.e. $\mathbf{R}^{-1} = \mathbf{R}^T$, $\det \mathbf{R} = +1$.

Noting that $f'(X) > 0$, $g'(Y) > 0$ as well as $r \doteq f(X) > 0$ have been imposed in eqn (1) to maintain the prescribed boundary correspondence, we then have $\det \mathbf{F} > 0$ and thus the polar decomposition of $\mathbf{F}$ given by

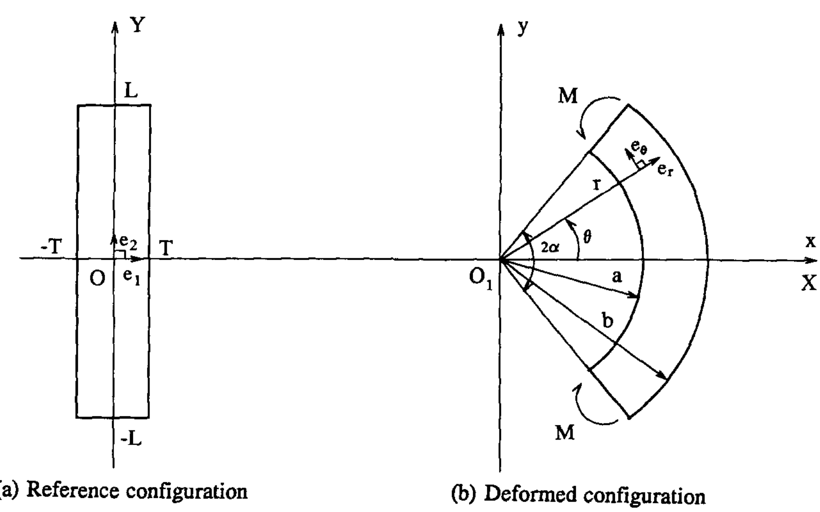

Fig. 1. Deformation geometry. (a) Reference configuration; (b) deformed configuration.

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR}. \tag{5}$$

The uniqueness of the decomposition then requires from eqns (3) and (5) that

$$\mathbf{U} = f'(X)\mathbf{e}_1 \otimes \mathbf{e}_1 + f(X)\,g'(Y)\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3. \tag{6}$$

This is the spectral representation of the right stretch tensor (of **F**). As a Lagrangian tensor, **U** is known to be the fundamental element of all Lagrangian strain tensors represented by the general form $(\mathbf{U}^m - \mathbf{I})/m$, with **I** the identity tensor and m any real number.

From eqns (4)–(6) it follows that

$$\mathbf{V} = f'(X)\mathbf{e}_r \otimes \mathbf{e}_r + f(X)\,g'(Y)\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \mathbf{k} \otimes \mathbf{k}. \tag{7}$$

This is the spectral representation of the left stretch tensor (of **F**). Similarly, it is known that **V**, as an Eulerian tensor, is the essential component of all Eulerian strain measures represented by $(\mathbf{V}^m - \mathbf{I})/m$. In particular, when $m = 0$, this yields, by a suitable limiting process, the result

$$\mathbf{H} = \ln \mathbf{V}, \tag{8}$$

which is known as the Hencky (logarithmic) strain. Since $\ln \mathbf{V}$ is coaxial with **V** and has principal values $\ln \lambda_i$ ($i = 1, 2, 3$), with λ_i the principal stretches of **V** (Ogden, 1984), we then have the general spectral representation of the Hencky strain measure as

$$\mathbf{H} = \sum_{i=1}^{3} (\ln \lambda_i)\mathbf{v}_i \otimes \mathbf{v}_i, \tag{9}$$

where $\mathbf{v}_i$ ($i = 1, 2, 3$) are the principal vectors of **V**. Being additive in measurements, identical in compression and tension tests and convenient in expressing the incompressibility condition (Mendelson, 1968), this strain measure is well known and has been widely used in metallurgical and material engineering, where it is regarded as the "natural strain" or "true strain" [see, for example, Nadai (1937)].

For the present problem, the Hencky strain is of the form

$$\mathbf{H} = \ln f'(X)\mathbf{e}_r \otimes \mathbf{e}_r + \ln\,[f(X)\,g'(Y)]\mathbf{e}_\theta \otimes \mathbf{e}_\theta, \tag{10}$$

which is obtained from eqns (7) and (9).

Since the principal vectors $\{\mathbf{e}_i,\ i = 1, 2, 3\}$ of **U** are essentially fixed in the present problem (see Fig. 1), it can be concluded [see Fitzgerald (1980); Gurtin and Spear (1983)] that the simple additivity of the Hencky strain measure exists in this case for arbitrarily large deformations. Consequently, this measure can effectively be applied to represent the additive total strain required to formulate the finite-strain deformation theory in the next section.

To measure the deformation of a material element using **H**, we require explicit expressions of the principal stretches λ_i. To this end, we assume that the material is incompressible. It then follows that

$$J \equiv \det \mathbf{F} = 1, \tag{11}$$

where J, as defined, is the local ratio of current to reference volume of a material volume element. Now using eqns (5) and (6) in eqn (11) yields

$$f'(X) f(X)\,g'(Y) = 1, \tag{12}$$

which implies that

$$f'(X)f(X) = \frac{1}{g'(Y)} = C, \tag{13}$$

with C a constant. This set of uncoupled ordinary differential equations, subject to the kinematic boundary conditions specified in eqn (1), can easily be solved. The solutions are

$$r^2 = f^2(X) = 2\frac{L}{\alpha}(X+T)+a^2, \quad \theta = g(Y) = \frac{\alpha}{L}Y, \tag{14}$$

where the undetermined constants a and α together with b are related by the constraint condition of global volume incompressibility, i.e.

$$\frac{b^2-a^2}{4T} = \frac{L}{\alpha}. \tag{15}$$

Now it follows from eqns (14) and (10) that the Hencky strain for the present problem is

$$\mathbf{H} = -\ln\left(\frac{\alpha}{L}r\right)\mathbf{e}_r \otimes \mathbf{e}_r + \ln\left(\frac{\alpha}{L}r\right)\mathbf{e}_\theta \otimes \mathbf{e}_\theta, \tag{16}$$

or in components

$$\varepsilon_r = \mathbf{e}_r \cdot \mathbf{H}\mathbf{e}_r = -\ln\left(\frac{\alpha}{L}r\right), \qquad \varepsilon_\theta = \mathbf{e}_\theta \cdot \mathbf{H}\mathbf{e}_\theta = \ln\left(\frac{\alpha}{L}r\right), \qquad \varepsilon_z = \mathbf{k} \cdot \mathbf{H}\mathbf{k} = 0, \tag{17}$$

where α, as a parameter, will be determined in Section 4 from the natural boundary conditions.

Once the parameter α is known, the strain field for any deformed configuration will readily be obtained from eqn (17). To determine α, however, we have to resort to the natural boundary conditions. Hence, we need to know the stress field first. To this end, we are required to find the constitutive equations which relate the material stress responses to the deformations produced by external factors (forces).

3. CONSTITUTIVE RELATIONS

For an elasto-plastic material there are two kinds of constitutive law, i.e. deformation theories and incremental theories. Since the plastic deformation of a work-hardening material is an irreversible process, the strain state of any material element in the plastically deformed configuration may depend on the entire loading history as well as on the final stress state of the element. Therefore, an incremental theory which relates increments of plastic strain to stress increments and stress history is required to represent such a plastic deformation. However, for the case of proportional loading an incremental theory can be reduced to a deformation theory by integrating along the loading path. Then deformation theories are valid for this particular case. For the present incompressible elasto-plastic plane strain problem with $\varepsilon_\theta+\varepsilon_r = 0$ and $\varepsilon_z = 0$, there exist the proportional relations $\varepsilon_\theta/\varepsilon_r = -1$, $\varepsilon_z/\varepsilon_\theta = 0 = \varepsilon_z/\varepsilon_r$ during the whole bending process. This implies that Hencky's deformation theory can effectively be employed to solve the present plane strain finite deformation problem.

Naturally, different kinds of material have different stress responses no matter which kind of constitutive law is employed. For our present purpose of deriving an exact analytical solution for the finite deformation elasto-plastic problem, a material with the true uniaxial

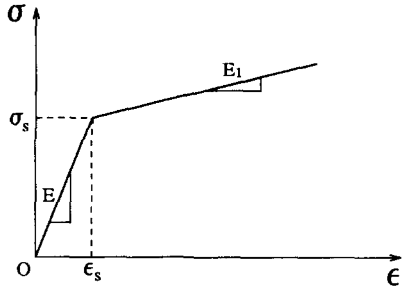

Fig. 2. True uniaxial stress–strain relation of the material.

stress–strain curve shown in Fig. 2 is adopted. This kind of material, which is different from the elastic power-law hardening material model used in Triantafyllidis (1980) as well as from the pure power-law hardening model applied in Chen (1962), is known as the elastic linear-hardening material. It may be characterized by

$$\sigma = \begin{cases} E\varepsilon, & \varepsilon \leqslant \varepsilon_s \\ \sigma_s + E_1(\varepsilon - \varepsilon_s), & \varepsilon > \varepsilon_s, \end{cases} \tag{18}$$

where σ_s, ε_s are the yield stress and strain, respectively, with $\varepsilon_s = \sigma_s/E$; E is Young's modulus; and E_1 is the tangent modulus. Experiments have shown that eqn (18) can be used to represent the stress–strain relations of many high-strength steels, such as modified 4330 steel (U.S.A.), 40CrNi3MoV steel (China) and 30SiMnMoVA steel (China) [see Gao (1993) for further references].

For complex stress and strain states, we may, on the basis of the "unique curve" hypothesis, modify eqn (18) to

$$\sigma_e = \begin{cases} E\varepsilon_e, & \varepsilon_e \leqslant \varepsilon_s \\ \sigma_s + E_1(\varepsilon_e - \varepsilon_s), & \varepsilon_e > \varepsilon_s, \end{cases} \tag{19}$$

where σ_e, ε_e are the effective stress and strain, respectively.

According to their definitions, an elastic–plastic material subjected to monotonic proportional loading, where deformation theories are applicable, is equivalent to a Cauchy elastic (hypoelastic) material, or a Green elastic (hyperelastic) material if it has smooth stress–strain curve which may be represented by a differentiable strain energy function (e.g. Ludwik-type pure power-law hardening elastic–plastic material). Hence, the deformation problem of a rate-independent elastic–plastic material (e.g. metals at room temperature) can be treated in a similar way to that of a non-linear elastic material. To form the constitutive relations for the finite deformation elasto-plastic problem in the context of finite elasticity theory using deformation theories of plasticity, we need an appropriate strain measure and its conjugate stress. As argued in the preceding section, the Hencky strain **H**, which retains its additivity for arbitrarily large deformations in this case, is adopted here as the desired strain measure. It has been shown by Hoger (1987) and Lehmann *et al.* (1991) that for an isotropic elastic material, where the Cauchy stress $\boldsymbol{\sigma}$ and the left stretch tensor **V** are coaxial, the stress conjugate to the logarithmic (Hencky) strain $\ln \mathbf{V}$ is the Kirchhoff stress $J\boldsymbol{\sigma}$. In particular, for an incompressible material with $J = 1$ this reduces to the Cauchy stress $\boldsymbol{\sigma}$. This is what is required for an Eulerian formulation of the boundary-value problem. Consequently, $\boldsymbol{\sigma}$ is adopted here as the conjugate stress of **H**.

For an isotropic incompressible elasto-plastic material with Poisson's ratio $\nu = 1/2$, the Hencky deformation theory for small strains may be expressed as [see, for example, Mendelson (1968)]

$$\varepsilon_{ij} = \frac{3}{2}\frac{\varepsilon_e}{\sigma_e}S_{ij}, \tag{20}$$

where S_{ij} is the deviatoric stress defined by

$$S_{ij} = \sigma_{ij} - p\delta_{ij} \tag{21}$$

with p the hydrostatic pressure and δ_{ij} the Kronecker delta. By using eqn (21), we can invert eqn (20) to give

$$\sigma_{ij} = \frac{2}{3}\frac{\sigma_e}{\varepsilon_e}\varepsilon_{ij} + p\delta_{ij}. \tag{22}$$

Now replace ε_{ij} by the Hencky strain $\mathbf{H}$, σ_{ij} by the Cauchy stress $\boldsymbol{\sigma}$ and $p\delta_{ij}$ by $-p\mathbf{I}$. Here the negative sign is introduced to ensure consistency with standard notation used in continuum mechanics texts. For example, in Truesdell (1977) the incompressibility constraint stress $\mathbf{N}$ is expressed as $\mathbf{N} = -p\mathbf{I}$ using an arbitrary Lagrangian multiplier $-p$, "the constraint pressure", rather than p, the hydrostatic pressure, defined in eqn (21). Consequently, we obtain the tensorial representation of the Hencky deformation theory for finite strains as

$$\boldsymbol{\sigma} = \frac{2}{3}\frac{\sigma_e}{\varepsilon_e}\mathbf{H} - p\mathbf{I}, \tag{23}$$

with the expressions of the effective stress σ_e and strain ε_e in this equation depending on the yield criterion as well as on the stress–strain curve to be used. Here the identity tensor $\mathbf{I}$ has the form

$$\mathbf{I} = \mathbf{e}_r \otimes \mathbf{e}_r + \mathbf{e}_\theta \otimes \mathbf{e}_\theta + \mathbf{k} \otimes \mathbf{k}. \tag{24}$$

Equations (23) and (19) consist of the constitutive relations required to solve this problem with ε_e (and/or σ_e) to be further specified.

When comparing eqn (23) with eqn (22), it is apparent that the only difference between these two different versions of Hencky's deformation theory is simply the Eulerian description vs Lagrangian description, though the Eulerian strain measure is the Hencky strain $\mathbf{H} = \ln \mathbf{V}$ rather than the most commonly used left Cauchy–Green strain $\mathbf{B} = \mathbf{V}^2$ or Almansi strain $\mathbf{E}_a = (\mathbf{I} - \mathbf{B}^{-1})/2$ (Ogden, 1984) in finite elasticity theory. The use of the Hencky strain $\mathbf{H}$ in component form in conjunction with Hencky's deformation theory for large deformation problems can be traced back to the early 1940s (MacGregor, 1948), where a closed-form finite deformation solution was obtained for a rigid–perfectly plastic closed-end (plane strain) cylinder. Recently, two exact finite deformation solutions for the elastic linear-hardening thick-walled cylinder (closed-end) and sphere were derived in Gao (1993, 1994), respectively, using the same theory and strain measure as those applied in MacGregor (1948), but in the Lagrangian formulation and with elastic large deformations included. The first systematic formulation for the finite-strain version of deformation theory, which was expressed in terms of the Lagrangian logarithmic strains and written in component form, was given by Hutchinson and Neale (1981), where several related theoretical problems were also discussed. This deformation theory was recently employed by Hou and Abeyaratne (1992) in tensor form.

4. FORMULATION OF THE BOUNDARY-VALUE PROBLEM

For a high-strength metal plate subjected to pure bending, as shown in Fig. 1(b), plastic yielding may start from the convex and/or concave surface and then expand inward to form two plastic zones. This happens only when the bending moments have reached certain critical values (called the elastic limit end moments in the next section) and continue to increase thereafter. Hence, we may assume that under some moments M the whole plate has already been finitely deformed to large curvature, whereas the plate material has only yielded near the convex and concave surfaces. From this point of view, it follows that the deformed plate may be divided into three adjacent domains: the large-deformation plastic domain I ($c_1 \leqslant r \leqslant b$), the elastic domain ($c_2 \leqslant r \leqslant c_1$) and the large-deformation plastic domain II ($a \leqslant r \leqslant c_2$) (see Fig. 3), which may be solved separately, as was done previously in Gao (1993, 1994) but in a different context. Here the two elasto-plastic interfaces are also taken to be cylindrical surfaces coaxial with the concave and convex cylindrical surfaces from symmetry, with c_1, c_2 the interface radii and p_{c_1}, p_{c_2} the associated interface pressures. Similarly, the neutral surface for which the fibre length remains unchanged is also a cylindrical surface of radius ρ_n coaxial with the two outer surfaces. To derive a general solution, we further assume that the neutral surface is within the elastic domain, i.e. $c_2 \leqslant \rho_n \leqslant c_1$, to allow for the purely elastic case.

Using the assumptions and analyses developed thus far, the finite deformation elasto-plastic problem can now be formulated as three separate boundary-value problems and solved analytically.

4.1. *Solution for plastic domain I* ($c_1 \leqslant r \leqslant b$)

This domain is within the positive side of the neutral surface (i.e. $\rho_n \leqslant c_1 \leqslant r \leqslant b$) in which fibres are momentarily extended, so that the strains in the domain are such that $\varepsilon_\theta \geqslant 0$ and $\varepsilon_r \leqslant 0$.

The basic equations, which embody the finite-strain version of Hencky's deformation theory and von Mises' yield criterion, are as follows: the equilibrium equations (no body forces)

$$\operatorname{div} \boldsymbol{\sigma} = \mathbf{0}, \tag{25}$$

the kinematic (geometrical) equations

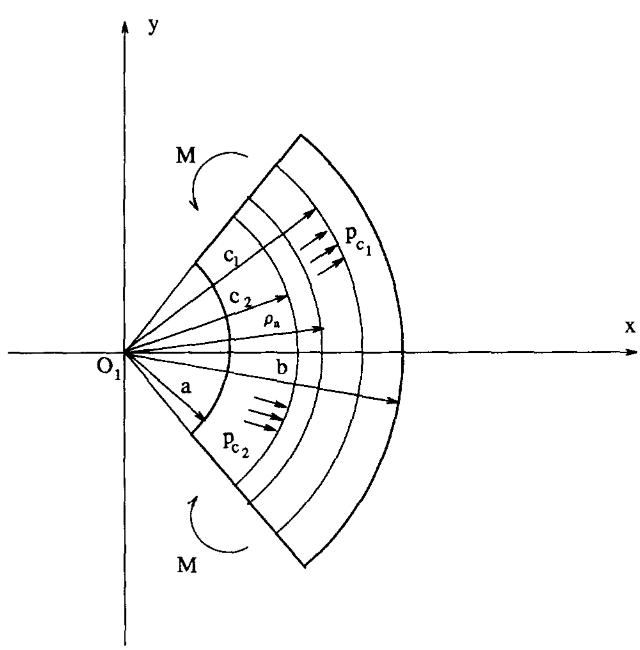

Fig. 3. Elastic and plastic domains of the deformed plate.

$$\mathbf{H} = \varepsilon_r \mathbf{e}_r \otimes \mathbf{e}_r + \varepsilon_\theta \mathbf{e}_\theta \otimes \mathbf{e}_\theta ;$$

$$\varepsilon_r = -\ln\left(\frac{\alpha}{L} r\right), \qquad \varepsilon_\theta = \ln\left(\frac{\alpha}{L} r\right), \tag{26a, b, c}$$

the constitutive equations

$$\boldsymbol{\sigma} = \frac{2}{3} \frac{\sigma_e}{\varepsilon_e} \mathbf{H} - p\mathbf{I}, \tag{27}$$

$$\sigma_e = \sigma_s + E_1(\varepsilon_e - \varepsilon_s), \tag{28}$$

where the effective strain ε_e, which is a positively increasing function of strain, is chosen to be associated with the von Mises yield criterion. For this case with $\varepsilon_z = 0$ and $\varepsilon_\theta = -\varepsilon_r > 0$, the effective strain is

$$\varepsilon_e = \frac{2}{\sqrt{3}} \varepsilon_\theta. \tag{29}$$

The boundary conditions are

$$\mathbf{t}(\mathbf{e}_r)|_{r=b} = \mathbf{0}, \qquad -\mathbf{t}(\mathbf{e}_r)|_{r=c_1} = p_{c_1} \mathbf{e}_r, \qquad \sigma_e|_{r=c_1} = \boldsymbol{\sigma}_s, \tag{30a, b, c}$$

where $\mathbf{t}$, defined by $\mathbf{t}(\mathbf{m}) = \boldsymbol{\sigma}\mathbf{m}$, is the traction vector associated with the normal vector $\mathbf{m}$.

Equations (25)–(30) define the determinate boundary-value problem in the plastic domain I ($c_1 \leqslant r \leqslant b$). It should be noted that no compatibility equations appear here. The reason for this is that the strain components in eqn (26), as logarithmic functions of the principal stretches λ_i, measure uniquely the deformation of the plate with the parameter α known (Hutchinson and Neale, 1981). This ensures that the strain components are themselves compatible.

Now this system of equations can be solved in the following manner. Substituting eqns (28), (29) into eqn (27) and using eqn (26c) will result in

$$\boldsymbol{\sigma} = \frac{2}{3}\left[E_1 + \frac{\sqrt{3}}{2}\sigma_s\left(1 - \frac{E_1}{E}\right)\ln^{-1}\left(\frac{\alpha}{L} r\right)\right]\mathbf{H} - p\mathbf{I}. \tag{31}$$

These are the constitutive relations of the present problem based on the finite-strain version of Hencky's deformation theory and von Mises yield criterion. Inserting eqn (31) into eqn (25) gives

$$\operatorname{grad} p = \frac{2}{3}\left[E_1 + \frac{\sqrt{3}}{2}\sigma_s\left(1 - \frac{E_1}{E}\right)\ln^{-1}\left(\frac{\alpha}{L} r\right)\right]\operatorname{div}\mathbf{H}$$
$$+ \mathbf{H}\operatorname{grad}\left\{\frac{2}{3}\left[E_1 + \frac{\sqrt{3}}{2}\sigma_s\left(1 - \frac{E_1}{E}\right)\ln^{-1}\left(\frac{\alpha}{L} r\right)\right]\right\}. \tag{32}$$

Substituting eqn (26) into this equation and carrying out the algebraic manipulations, we finally obtain

$$\frac{\mathrm{d}p}{\mathrm{d}r} = \frac{2}{3}\left[E_1 + \frac{\sqrt{3}}{2}\sigma_s\left(1 - \frac{E_1}{E}\right)\ln^{-1}\left(\frac{\alpha}{L} r\right)\right]$$
$$\times\left[-\frac{1}{r} - \frac{2}{r}\ln\left(\frac{\alpha}{L} r\right)\right] + \frac{1}{\sqrt{3}}\sigma_s\left(1 - \frac{E_1}{E}\right)\ln^{-1}\left(\frac{\alpha}{L} r\right)\frac{1}{r}. \tag{33}$$

This equation may be regarded as the basic governing equation for the boundary-value problem in question.

Integrating eqn (33) from b to r gives the hydrostatic (constraint) pressure as

$$p(r) = p(b) - \frac{2}{3}\left\{E_1\left[1+\ln\left(\frac{\alpha^2}{L^2}rb\right)\right]+\sqrt{3}\sigma_s\left(1-\frac{E_1}{E}\right)\right\}\ln\frac{r}{b}, \tag{34}$$

where $p(b)$ is determined from eqns (30a) and (31) as

$$p(b) = -\frac{1}{\sqrt{3}}\sigma_s\left(1-\frac{E_1}{E}\right)-\frac{2}{3}E_1\ln\left(\frac{\alpha}{L}b\right). \tag{35}$$

With $p(r)$ known, it then follows from eqns (31) and (26) that the Cauchy stress is

$$\boldsymbol{\sigma} = \frac{2}{3}\left[E_1+\frac{\sqrt{3}}{2}\sigma_s\left(1-\frac{E_1}{E}\right)\ln^{-1}\left(\frac{\alpha}{L}r\right)\right]\left[-\ln\left(\frac{\alpha}{L}r\right)\mathbf{e}_r\otimes\mathbf{e}_r+\ln\left(\frac{\alpha}{L}r\right)\mathbf{e}_\theta\otimes\mathbf{e}_\theta\right]-p(r)\mathbf{I}, \tag{36}$$

or in component form

$$\begin{aligned}\sigma_r &= \mathbf{e}_r\cdot\boldsymbol{\sigma}\mathbf{e}_r = -\frac{1}{\sqrt{3}}\sigma_s\left(1-\frac{E_1}{E}\right)-\frac{2}{3}E_1\ln\left(\frac{\alpha}{L}r\right)-p(r),\\ \sigma_\theta &= \mathbf{e}_\theta\cdot\boldsymbol{\sigma}\mathbf{e}_\theta = \frac{1}{\sqrt{3}}\sigma_s\left(1-\frac{E_1}{E}\right)+\frac{2}{3}E_1\ln\left(\frac{\alpha}{L}r\right)-p(r),\\ \sigma_z &= \mathbf{k}\cdot\boldsymbol{\sigma}\mathbf{k} = -p(r),\end{aligned} \tag{37}$$

where $p(r)$ is given in eqn (34).

Equations (36) and (37) together with eqns (16) and (17) represent the general solution for the plastic domain I ($c_1 \leqslant r \leqslant b$), with α and b acting as undetermined parameters. To determine $\boldsymbol{\sigma}$ and $\mathbf{H}$, we still need to find the parameters in terms of known geometrical quantities L, T, and W, and material constants σ_s, E, and E_1. These relations can be derived from the natural boundary conditions.

Equation (30c) together with eqns (28), (29), and (26c) gives

$$\frac{\alpha}{L}c_1 = \exp\left(\frac{\sqrt{3}}{2}\varepsilon_s\right), \tag{38}$$

which establishes the first relation required.

The use of eqn (30b) in conjunction with eqn (36) yields

$$p_{c_1} = -\frac{2}{3}\left[\sqrt{3}\sigma_s\left(1-\frac{E_1}{E}\right)+E_1\ln\left(\frac{\alpha^2}{L^2}c_1b\right)\right]\ln\frac{c_1}{b}. \tag{39}$$

After eliminating p_{c_1} this equation provides another relation to solve for the unknown parameters. To do this, however, we need to know the adjacent elastic domain solution as well as the plastic domain II solution.

4.2. *Solution for plastic domain II* ($a \leqslant r \leqslant c_2$)

This domain is within the negative side of the neutral surface (i.e. $a \leqslant r \leqslant c_2 \leqslant \rho_n$) in which fibres are momentarily compressed, so that the strains in the domain are such that

$\varepsilon_\theta \leqslant 0$ and $\varepsilon_r \geqslant 0$. This will result in an expression of the effective strain different from that of the plastic domain I, but all of the other governing equations remain the same.

Now using the effective strain associated with the von Mises yield criterion, i.e.

$$\varepsilon_e = -\frac{2}{\sqrt{3}}\varepsilon_\theta, \tag{40}$$

and the boundary conditions

$$-\mathbf{t}(\mathbf{e}_r)|_{r=a} = \mathbf{0}, \qquad \mathbf{t}(\mathbf{e}_r)|_{r=c_2} = -p_{c_2}\mathbf{e}_r, \qquad \sigma_e|_{r=c_2} = \sigma_s \tag{41a, b, c}$$

to replace eqns (29) and (30), respectively, the boundary-value problem for determining the solution for the plastic domain II ($a \leqslant r \leqslant c_2$) will be formed with eqns (25)–(28) and (40), (41). This problem can be solved by using the same procedures as for the boundary-value problem in the plastic domain I.

The solution gives the Cauchy stress as

$$\boldsymbol{\sigma} = \frac{2}{3}\left[E_1 - \frac{\sqrt{3}}{2}\sigma_s\left(1-\frac{E_1}{E}\right)\ln^{-1}\left(\frac{\alpha}{L}r\right)\right]\left[-\ln\left(\frac{\alpha}{L}r\right)\mathbf{e}_r\otimes\mathbf{e}_r + \ln\left(\frac{\alpha}{L}r\right)\mathbf{e}_\theta\otimes\mathbf{e}_\theta\right] - p(r)\mathbf{I}, \tag{42}$$

or in components

$$\begin{aligned}
\sigma_r &= \frac{1}{\sqrt{3}}\sigma_s\left(1-\frac{E_1}{E}\right) - \frac{2}{3}E_1\ln\left(\frac{\alpha}{L}r\right) - p(r),\\
\sigma_\theta &= -\frac{1}{\sqrt{3}}\sigma_s\left(1-\frac{E_1}{E}\right) + \frac{2}{3}E_1\ln\left(\frac{\alpha}{L}r\right) - p(r),\\
\sigma_z &= -p(r),
\end{aligned} \tag{43}$$

where the hydrostatic (constraint) pressure $p(r)$ is

$$p(r) = \frac{1}{\sqrt{3}}\sigma_s\left(1-\frac{E_1}{E}\right) - \frac{2}{3}E_1\ln\left(\frac{\alpha}{L}a\right) + \frac{2}{3}\left\{\sqrt{3}\sigma_s\left(1-\frac{E_1}{E}\right) - E_1\left[1+\ln\left(\frac{\alpha^2}{L^2}ra\right)\right]\right\}\ln\frac{r}{a}, \tag{44}$$

where use of the boundary condition, eqn (41a), has been made.

Equations (42)–(44) together with eqns (16), (17) represent the general solution for the plastic domain II ($a \leqslant r \leqslant c_2$), with α and a acting as parameters to be determined.

Now apply the remaining two boundary conditions of this boundary-value problem.

From eqns (41c), (28), (40), and (26c), it follows that

$$\frac{\alpha}{L}c_2 = \exp\left(-\frac{\sqrt{3}}{2}\varepsilon_s\right), \tag{45}$$

which gives the second relation required.

Substituting eqns (42) and (44) into eqn (41b) results in

$$p_{c_2} = \frac{2}{3}\left[\sqrt{3}\sigma_s\left(1-\frac{E_1}{E}\right)-E_1\ln\left(\frac{\alpha^2}{L^2}c_2a\right)\right]\ln\frac{c_2}{a}. \tag{46}$$

This expression for the interface pressure p_{c_2} will be used in deriving the elastic domain solution.

4.3. *Solution for the elastic domain* ($c_2 \leqslant r \leqslant c_1$)

Using the same basic assumptions as those used to determine the solution for the plastic domain I, and the following constitutive equations

$$\boldsymbol{\sigma} = \tfrac{2}{3}E\mathbf{H}-p\mathbf{I}, \tag{47}$$

$$\sigma_e = E\varepsilon_e, \tag{48}$$

and the boundary conditions

$$-\mathbf{t}(\mathbf{e}_r)|_{r=c_2} = p_{c_2}\mathbf{e}_r, \qquad \mathbf{t}(\mathbf{e}_r)|_{r=c_1} = -p_{c_1}\mathbf{e}_r \tag{49a, b}$$

to replace eqns (27), (28), and (30), respectively, the boundary-value problem to determine the solution for the elastic domain ($c_2 \leqslant r \leqslant c_1$) will be formed with eqns (25), (26) and (47)–(49). This problem can also be solved by following procedures similar to those used for solving the boundary-value problem in the plastic domain I.

The solution gives the Cauchy stress as

$$\boldsymbol{\sigma} = \frac{2}{3}E\left[-\ln\left(\frac{\alpha}{L}r\right)\mathbf{e}_r\otimes\mathbf{e}_r+\ln\left(\frac{\alpha}{L}r\right)\mathbf{e}_\theta\otimes\mathbf{e}_\theta\right]-p(r)\mathbf{I}, \tag{50}$$

or in components

$$\sigma_r = -\frac{2}{3}E\ln\left(\frac{\alpha}{L}r\right)-p(r), \qquad \sigma_\theta = \frac{2}{3}E\ln\left(\frac{\alpha}{L}r\right)-p(r), \qquad \sigma_z = -p(r), \tag{51}$$

where the hydrostatic (constraint) pressure $p(r)$ is

$$p(r) = \frac{2}{3}\left[\sqrt{3}\sigma_s\left(1-\frac{E_1}{E}\right)-E_1\ln\left(\frac{\alpha^2}{L^2}c_2a\right)\right]\ln\frac{c_2}{a} - \frac{2}{3}E\left\{\ln\left(\frac{\alpha}{L}c_2\right)+\left[1+\ln\left(\frac{\alpha^2}{L^2}rc_2\right)\right]\ln\frac{r}{c_2}\right\}, \tag{52}$$

where use has been made of the boundary condition in eqn (49a) as well as of eqn (46).

Equations (50)–(52) together with eqns (16), (17) represent the general solution for the elastic domain ($c_2 \leqslant r \leqslant c_1$), with a, α, and c_2 as the parameters to be determined from the natural boundary conditions.

Substituting eqns (50) and (52) into eqn (49b) will yield

$$p_{c_1} = \frac{2}{3}E\ln\left(\frac{\alpha}{L}c_1\right)+\frac{2}{3}\left[\sqrt{3}\sigma_s\left(1-\frac{E_1}{E}\right)-E_1\ln\left(\frac{\alpha^2}{L^2}c_2a\right)\right]\ln\frac{c_2}{a} - \frac{2}{3}E\left\{\ln\left(\frac{\alpha}{L}c_2\right)+\left[1+\ln\left(\frac{\alpha^2}{L^2}c_1c_2\right)\right]\ln\frac{c_1}{c_2}\right\}. \tag{53}$$

Then eliminating p_{c_1} from eqns (39) and (53) results in

$$\sqrt{3}\sigma_s\left(1-\frac{E_1}{E}\right)\left(\ln\frac{b}{c_1}-\ln\frac{c_2}{a}\right)+E_1\left[\ln\left(\frac{\alpha^2}{L^2}bc_1\right)\ln\frac{b}{c_1}+\ln\left(\frac{\alpha^2}{L^2}c_2a\right)\ln\frac{c_2}{a}\right]=-E\ln\left(\frac{\alpha^2}{L^2}c_1c_2\right)\ln\frac{c_1}{c_2}. \quad (54)$$

This equation provides the third relation to determine α, a, b, c_1, and c_2.

The incompressibility constraint condition given in eqn (15) furnishes a fourth relation.

The fifth and last equation for determining the unknown parameters is derived from the only remaining global natural boundary condition given by

$$\int_\Sigma [\mathbf{r}\times\mathbf{t}(\mathbf{e}_\theta)]|_{\theta=\alpha}\,\mathrm{d}A = M\mathbf{k}, \quad (55)$$

where Σ denotes the area of the end cross-section: $a\leqslant r\leqslant b$, $-W\leqslant z\leqslant W$, $\theta=\alpha$; M represents the end moments (i.e. the moment couple) required to maintain the specified deformation; and $\mathbf{r} = r\mathbf{e}_r(\theta)+z\mathbf{k}$ is the position vector of a material point relative to O_1 in Fig. 1(b).

Now substituting eqns (36), (42), and (50) into eqn (55) gives

$$\int_a^{c_2}\left[-\frac{1}{\sqrt{3}}\sigma_s\left(1-\frac{E_1}{E}\right)+\frac{2}{3}E_1\ln\left(\frac{\alpha}{L}r\right)-p_{P_{II}}(r)\right]\mathrm{d}r+\int_{c_2}^{c_1}\left[\frac{2}{3}E\ln\left(\frac{\alpha}{L}r\right)-p_E(r)\right]\mathrm{d}r+\int_{c_1}^{b}\left[\frac{1}{\sqrt{3}}\sigma_s\left(1-\frac{E_1}{E}\right)+\frac{2}{3}E_1\ln\left(\frac{\alpha}{L}r\right)-p_{P_I}(r)\right]\mathrm{d}r=0,$$

$$\int_a^{c_2}r\left[-\frac{1}{\sqrt{3}}\sigma_s\left(1-\frac{E_1}{E}\right)+\frac{2}{3}E_1\ln\left(\frac{\alpha}{L}r\right)-p_{P_{II}}(r)\right]\mathrm{d}r+\int_{c_2}^{c_1}r\left[\frac{2}{3}E\ln\left(\frac{\alpha}{L}r\right)-p_E(r)\right]\mathrm{d}r+\int_{c_1}^{b}r\left[\frac{1}{\sqrt{3}}\sigma_s\left(1-\frac{E_1}{E}\right)+\frac{2}{3}E_1\ln\left(\frac{\alpha}{L}r\right)-p_{P_I}(r)\right]\mathrm{d}r=\frac{M}{2W}, \quad (56a, b)$$

where $p_{P_{II}}(r)$, $p_E(r)$, and $p_{P_I}(r)$ are given by eqns (44), (52), and (34), respectively. After some lengthy but straightforward algebra, eqn (56a) turns out to be

$$c_1\left\{\sqrt{3}\sigma_s\left(1-\frac{E_1}{E}\right)\left(\ln\frac{b}{c_1}-\ln\frac{c_2}{a}\right)+E_1\left[\ln\left(\frac{\alpha^2}{L^2}bc_1\right)\ln\frac{b}{c_1}+\ln\left(\frac{\alpha^2}{L^2}c_2a\right)\ln\frac{c_2}{a}\right]+E\ln\left(\frac{\alpha^2}{L^2}c_1c_2\right)\ln\frac{c_1}{c_2}\right\}=0, \quad (57)$$

while eqn (56b) yields

$$\begin{aligned}\frac{M}{2W}=&\frac{1}{3}\sqrt{3}\sigma_s\left(1-\frac{E_1}{E}\right)\left\{c_1^2\left(\ln\frac{b}{c_1}-\ln\frac{c_2}{a}\right)+\frac{1}{2}[(b^2-c_1^2)-(c_2^2-a^2)]\right\}\\&+\frac{1}{3}E_1\left\{\left[b^2\ln\left(\frac{\alpha}{L}b\right)-c_1^2\ln\left(\frac{\alpha}{L}c_1\right)\right]+\left[c_2^2\ln\left(\frac{\alpha}{L}c_2\right)-a^2\ln\left(\frac{\alpha}{L}a\right)\right]\right.\\&\left.+c_1^2\left[\ln\left(\frac{\alpha^2}{L^2}bc_1\right)\ln\frac{b}{c_1}+\ln\left(\frac{\alpha^2}{L^2}c_2a\right)\ln\frac{c_2}{a}\right]-\frac{1}{2}[(b^2-c_1^2)+(c_2^2-a^2)]\right\}\\&+\frac{1}{3}E\left\{c_1^2\ln\left(\frac{\alpha^2}{L^2}c_1c_2\right)\ln\frac{c_1}{c_2}+\left[c_1^2\ln\left(\frac{\alpha}{L}c_1\right)-c_2^2\ln\left(\frac{\alpha}{L}c_2\right)\right]-\frac{1}{2}(c_1^2-c_2^2)\right\}. \quad (58)\end{aligned}$$

Obviously, since $c_1 > 0$, eqn (57) is identically satisfied as long as eqn (54) is met. This result is actually a physical consequence of the traction-free boundary conditions which have been used in eqns (30a) and (41a), as was shown by Hill (1950) for a fully yielded rigid–perfectly plastic plate in pure bending. Consequently, eqn (58) alone provides the fifth relation needed to determine the unknown parameters.

Combining eqns (15), (38), (45), (54), and (58) leads to the system of equations for determining the unknown parameters α, a, b, c_1, and c_2. This system may be written in the form

$$\frac{\alpha}{L}c_1 = \exp\left(\frac{\sqrt{3}}{2}\varepsilon_s\right),$$

$$\frac{\alpha}{L}c_2 = \exp\left(-\frac{\sqrt{3}}{2}\varepsilon_s\right),$$

$$\frac{b^2-a^2}{4T} = \frac{L}{\alpha},$$

$$ab = c_1c_2,$$

$$\frac{M}{2W} = \text{right-hand side of eqn (58)}, \qquad (59\text{a, b, c, d, e})$$

where the fourth equation is the simplified version of eqn (54) resulting from the substitution of eqns (38) and (45) [i.e. eqns (59a,b)]. From eqn (59d) it is clear that the convex and concave surfaces yield at the same time, i.e. $c_2 = a$ whenever $c_1 = b$. This differs from what was observed in Shaffer and House (1955) for an elastic–perfectly plastic wide curved plate undergoing small deformations, but it can easily be shown that the present result agrees with the physical deduction given in Triantafyllidis (1980) [see his eqn (2.12)] for an elastic power-law hardening wide plate in the presence of large deformations. It may therefore be concluded that this is a general result for all plane strain wide plates of elastic–strain-hardening–plastic materials subjected to finite deformation pure bending. By solving this set of non-linear algebraic equations, we obtain the values of α, a, b, c_1, and c_2. The solutions of this system, which are obtained by taking $\phi = c_2/a$ as known *a priori*, are

$$a = 4T\frac{\phi\exp\left(\frac{\sqrt{3}}{2}\varepsilon_s\right)}{\phi^4\exp(2\sqrt{3}\varepsilon_s)-1}, \qquad b = 4T\frac{\phi^3\exp\left(\frac{3}{2}\sqrt{3}\varepsilon_s\right)}{\phi^4\exp(2\sqrt{3}\varepsilon_s)-1},$$

$$\frac{\alpha}{L} = \frac{1}{4T}\frac{\phi^4\exp(2\sqrt{3}\varepsilon_s)-1}{\phi^2\exp(\sqrt{3}\varepsilon_s)}, \qquad c_1 = 4T\frac{\phi^2\exp\left(\frac{3}{2}\sqrt{3}\varepsilon_s\right)}{\phi^4\exp(2\sqrt{3}\varepsilon_s)-1},$$

$$c_2 = \phi a = 4T\frac{\phi^2\exp\left(\frac{\sqrt{3}}{2}\varepsilon_s\right)}{\phi^4\exp(2\sqrt{3}\varepsilon_s)-1}, \qquad (60)$$

where ϕ is chosen in such a way that eqn (59e) is satisfied by the parameters a, b, α, c_1, and c_2 so determined. For given end moments M as well as known geometrical constants T, L, W and material properties σ_s, E, and E_1, ϕ may be found by solving the equation resulting from substituting eqn (60) into eqn (59e) using an iteration procedure (e.g. the Newton–Raphson method). On the other hand, if one of these parameters is given, then ϕ as well as the other parameters can be determined from eqn (60), i.e. the deformation becomes known. Alternatively, eqn (59e) can then be used to calculate the end moments

M required to maintain this deformation. Some numerical results will be given in the next section using the latter approach.

For the deformation associated with the end moments M, determined from eqns (59e) and (60), the corresponding stress distributions in the plastic domain I ($c_1 \leqslant r \leqslant b$), the elastic domain ($c_2 \leqslant r \leqslant c_1$), and the plastic domain II ($a \leqslant r \leqslant c_2$) can easily be calculated from eqns (36) and (37), eqns (50)–(52), and eqns (42)–(44), respectively. The strain distributions in all of the three domains can be determined from eqns (16) and (17).

5. ANALYSES AND APPLICATIONS OF THE SOLUTION

From the analytical expressions of the general solution obtained in the preceding section, some specific solutions may now be derived. Numerical results are also provided here to show the applications of the solution.

5.1. *Solution for an elastic–perfectly plastic wide plate*

When $E_1 = 0$, eqn (18) characterizes the stress–strain relation of an elastic–perfectly plastic material.

Let E_1 in eqn (36) be equal to zero; then, the Cauchy stress in the plastic domain I is found to be

$$\boldsymbol{\sigma} = \frac{1}{\sqrt{3}}\sigma_s(-\mathbf{e}_r \otimes \mathbf{e}_r + \mathbf{e}_\theta \otimes \mathbf{e}_\theta) - p(r)\mathbf{I}, \tag{61}$$

where $p(r)$ is obtained from eqn (34) with $E_1 = 0$ as

$$p(r) = -\frac{1}{\sqrt{3}}\sigma_s\left(1 + 2\ln\frac{r}{b}\right). \tag{62}$$

In component form, eqn (61) becomes

$$\sigma_r = \frac{2}{\sqrt{3}}\sigma_s \ln\frac{r}{b}, \qquad \sigma_\theta = \frac{2}{\sqrt{3}}\sigma_s\left(1 + \ln\frac{r}{b}\right), \qquad \sigma_z = \frac{1}{\sqrt{3}}\sigma_s\left(1 + 2\ln\frac{r}{b}\right). \tag{63}$$

Similarly, the Cauchy stress in the plastic domain II follows from eqn (42) with $E_1 = 0$ as

$$\boldsymbol{\sigma} = \frac{1}{\sqrt{3}}\sigma_s[\mathbf{e}_r \otimes \mathbf{e}_r - \mathbf{e}_\theta \otimes \mathbf{e}_\theta] - p(r)\mathbf{I}, \tag{64}$$

where $p(r)$ is obtained from eqn (44) with $E_1 = 0$ as

$$p(r) = \frac{1}{\sqrt{3}}\sigma_s\left(1 + 2\ln\frac{r}{a}\right). \tag{65}$$

If expressed in component form, eqn (64) gives

$$\sigma_r = -\frac{2}{\sqrt{3}}\sigma_s \ln\frac{r}{a}, \qquad \sigma_\theta = -\frac{2}{\sqrt{3}}\sigma_s\left(1 + \ln\frac{r}{a}\right), \qquad \sigma_z = -\frac{1}{\sqrt{3}}\sigma_s\left(1 + 2\ln\frac{r}{a}\right). \tag{66}$$

In the above equations b, a together with α, c_1 and c_2 are given by eqn (60), with ϕ determined by the following equation resulting from eqn (59e) with $E_1 = 0$:

$$\frac{M}{2W}=\frac{1}{3}\sqrt{3}\sigma_s\left\{c_1^2\left(\ln\frac{b}{c_1}-\ln\frac{c_2}{a}\right)+\frac{1}{2}[(b^2-c_1^2)-(c_2^2-a^2)]\right\}$$

$$+\frac{1}{3}E\left\{c_1^2\ln\left(\frac{\alpha^2}{L^2}c_1c_2\right)\ln\frac{c_1}{c_2}+\left[c_1^2\ln\left(\frac{\alpha}{L}c_1\right)-c_2^2\ln\left(\frac{\alpha}{L}c_2\right)\right]-\frac{1}{2}(c_1^2-c_2^2)\right\},\quad (67)$$

where a, b, α, c_1, and c_2 are to be substituted from eqn (60).

With α, a, and c_2 so determined, it then follows from eqns (50) and (51) that the Cauchy stress in the elastic domain is

$$\boldsymbol{\sigma}=\frac{2}{3}E\left[-\ln\left(\frac{\alpha}{L}r\right)\mathbf{e}_r\otimes\mathbf{e}_r+\ln\left(\frac{\alpha}{L}r\right)\mathbf{e}_\theta\otimes\mathbf{e}_\theta\right]-p(r)\mathbf{I},\quad (68)$$

or in component form

$$\sigma_r=-\frac{2}{3}E\ln\left(\frac{\alpha}{L}r\right)-p(r),\qquad \sigma_\theta=\frac{2}{3}E\ln\left(\frac{\alpha}{L}r\right)-p(r),\qquad \sigma_z=-p(r),\quad (69)$$

where the hydrostatic (constraint) pressure $p(r)$ is obtained from eqn (52) with $E_1=0$ as

$$p(r)=\frac{2}{\sqrt{3}}\sigma_s\ln\frac{c_2}{a}-\frac{2}{3}E\left\{\ln\left(\frac{\alpha}{L}c_2\right)+\left[1+\ln\left(\frac{\alpha^2}{L^2}rc_2\right)\right]\ln\frac{r}{c_2}\right\}.\quad (70)$$

Equations (61)–(70) together with eqn (60) represent an exact finite deformation elasto-plastic solution for an elastic–perfectly plastic wide plate subjected to pure bending. In particular, when $c_1=c_2=c=\rho_n$, i.e. the limiting case of fully yielded plate, this specific solution reduces to the fully yielded solution given by Hill (1950) for a rigid–perfectly plastic wide plate. However, no deformation analysis was included in Hill (1950), where the parameters a, b, and α were simply regarded as known *a priori* instead of determining them from both the kinematic and natural boundary conditions. It is of particular interest to note that when $c_1=c_2=c$ eqn (67) reduces to

$$\frac{M}{2W}=\frac{2}{\sqrt{3}}\sigma_s\frac{1}{4}(b-a)^2,\quad (71)$$

which is identical with that given in Hill (1950). However, it should be pointed out that since the current plate thickness $(b-a)$ is, in general, not equal to the initial thickness $2T$, we are not entitled to say that the end moments M are independent of the amount of the plastic bending as was concluded in Hill (1950), which might be only appropriate for small deformations.

5.2. *Solution for an elastic linear-hardening wide plate under its plastic limit end moments*

When the extent of the elastic domain ($c_2\leqslant r\leqslant c_1$) near the neutral surface becomes negligibly small under sufficiently large end moments, we may regard the plate to be in a fully yielded state with $c_1=c_2$. Consequently, the end moments withstood by the plate in such a state are called the plastic limit end moments of the finitely deformed plate, denoted by $M^p_{\max}$.

Let $c_1=c_2=c$ in eqn (59); then

$$\frac{\alpha}{L}c = 1, \qquad \frac{b^2-a^2}{4T} = \frac{L}{\alpha}, \qquad ab = c^2, \tag{72a, b, c}$$

with a, b, α, and c as well as $M^p_{\max}$ satisfying the following equation obtained from eqn (59e) with $c_1 = c_2 = c$:

$$\frac{M^p_{\max}}{2W} = \frac{1}{6}\sqrt{3}\sigma_s\left(1-\frac{E_1}{E}\right)(b-a)^2+\frac{1}{3}E_1\left[b^2\ln\left(\frac{\alpha}{L}b\right)-a^2\ln\left(\frac{\alpha}{L}a\right)-\frac{1}{2}(b^2-a^2)\right], \tag{73}$$

where use has been made of eqns (72a) and (72c).

Equation (72a) implies that the neutral surface with $\varepsilon_\theta = 0$ is identical with the elasto-plastic interface in this limiting case, i.e. $c = \rho_n$ for the fully yielded plate in pure bending. Clearly, since eqn (72a) is a purely kinematic consequence, this result is valid for any finitely deformed wide plate subjected to pure bending, i.e. there are no restrictions on the material model used. This may verify and generalize the hypothesis used by Hill (1950) for a rigid–perfectly plastic plate. Equation (72c) yields $c = (ab)^{1/2}$, which was also used in Hill (1950). Again, this relation is general, and not specific to plates of rigid–perfectly plastic material.

Similarly, by taking $\phi = c/a$ as known *a priori*, eqns (72) and (73) can be solved to give

$$a = 4T\frac{\phi}{\phi^4-1}, \qquad b = 4T\frac{\phi^3}{\phi^4-1}, \qquad \frac{\alpha}{L} = \frac{1}{4T}(\phi^2-\phi^{-2}), \qquad c = 4T\frac{\phi^2}{\phi^4-1}, \tag{74}$$

where ϕ ($\phi \neq 1$) is determined from eqn (73) after substituting eqn (74).

With the parameters determined, the stress and strain fields within the fully yielded finite deformation plate ($a \leqslant r \leqslant b$) will readily be obtained from eqns (36), (37) and (16), (17) for $c \leqslant r \leqslant b$ and from eqns (42)–(44) and (16), (17) for $a \leqslant r \leqslant c$, respectively.

5.3. *Solution for an elastic linear-hardening wide plate under its elastic limit end moments*

When $c_1 = b$, i.e. the convex surface of the plate yields, the end moments withstood by the plate are called the elastic limit end moments of the finitely deformed plate, denoted by $M^e_{\max}$. Under these moments the concave surface also yields, i.e. $c_2 = a$ simultaneously, as argued previously.

Let c_1 and c_2 in eqns (59a–c) be equal to b and a, respectively; then

$$\frac{\alpha}{L}b = \exp\left(\frac{\sqrt{3}}{2}\varepsilon_s\right), \qquad \frac{\alpha}{L}a = \exp\left(-\frac{\sqrt{3}}{2}\varepsilon_s\right), \qquad \frac{b^2-a^2}{4T} = \frac{L}{\alpha}. \tag{75a, b, c}$$

This system of equations can easily be solved to give

$$a = 4T\frac{\exp\left(\frac{\sqrt{3}}{2}\varepsilon_s\right)}{\exp(2\sqrt{3}\varepsilon_s)-1}, \qquad b = 4T\frac{\exp\left(\frac{3}{2}\sqrt{3}\varepsilon_s\right)}{\exp(2\sqrt{3}\varepsilon_s)-1},$$

$$\frac{\alpha}{L} = \frac{1}{4T}[\exp(\sqrt{3}\varepsilon_s)-\exp(-\sqrt{3}\varepsilon_s)]. \tag{76}$$

Then it follows from eqn (59e) with $c_1 = b$ and $c_2 = a$ that the elastic limit moments are

Table 1. End moments for various specified deformations

$2T$	σ_s/E	E_1/E	ρ_n	a	b	c_1	c_2	α/L	ϕ	$b-a$	$2\pi L/\alpha$	$M/2W\sigma_s$
1.0	0.002	0.0	2.5	2.057	3.038	2.504	2.496	0.4	1.213	0.981	15.71	0.278
		0.2										4.406
		0.5										10.60
		0.8										16.79
1.0	0.002	0.0	5.0	4.527	5.522	5.009	4.991	0.2	1.103	0.995	31.42	0.286
		0.2										2.417
		0.5										5.612
		0.8										8.807
2.0	0.002	0.0	2.5	1.733	3.606	2.504	2.496	0.4	1.440	1.873	15.71	1.013
		0.2										29.62
		0.5										72.52
		0.8										115.4
2.0	0.002	0.0	5.0	4.114	6.077	5.009	4.991	0.2	1.213	1.963	31.42	1.112
		0.2										17.62
		0.5										42.39
		0.8										67.16

$$\frac{M^e_{\max}}{2W} = \frac{1}{6}[\sqrt{3}\sigma_s(b^2+a^2) - E(b^2-a^2)], \tag{77}$$

where use has been made of eqns (75a) and (75b). Obviously, by definition, when $M > M^e_{\max}$ the finitely deformed plate will be in an elastic–partially plastic state.

With the parameters known, the stress and strain fields within the finitely deformed elastic plate ($a \leqslant r \leqslant b$) will easily be determined from eqns (50)–(52) and (16), (17).

Since this is a purely elastic solution with only the two outer surfaces yielded, it is also applicable to elastic non-linear-hardening plates undergoing large deformations. Moreover, it is clear that this solution is a closed-form one.

If we set $\alpha = 2\pi$ and view α/L instead of α as a parameter in the expressions derived so far, the general solution as well as the resulting specific solutions will represent the important case of the bending of a flat plate into a cylindrical shell. Consequently, the initial length L of the plate required to form such a cylindrical shell will be obtained from the parameter finally determined. This is of particular value in pressure vessel technology.

Some numerical results are listed in Table 1 to show quantitatively the applications of the solution.

In Table 1, T, a, b, c_1, c_2, $b-a$, ρ_n and L/α have dimensions of length, α/L has dimension of $(\text{length})^{-1}$, $M/2W\sigma_s$ has dimension of $(\text{length})^2$ while σ_s/E, E_1/E and ϕ are non-dimensional. These numerical values were obtained from eqns (59e) and (60) using a simple PC computer code. The results listed under the column heading "$2\pi L/\alpha$" are the values of the initial plate length required for forming cylinders of different neutral radii ρ_n.

As expected, the results in Table 1 show that the moment capacities of the plates with the same deformations are strongly dependent on the strain-hardening effects of material: the larger E_1/E, the larger $M/2W\sigma_s$. It is also shown that different bending moments are required for the plates with different values of thickness, which inherently correspond to different deformations, to obtain the same neutral radius ρ_n: the larger $2T$, the larger $M/2W\sigma_s$. When comparing the values of $M/2W\sigma_s$ corresponding to the elastic–perfectly plastic material plates ($E_1 = 0$) with the same initial thickness $2T$, we find that they are different for different plastic deformations (i.e. ϕ is different), which result in different values of the current thickness $b-a$. This implies that the end moments are dependent on the amount of plastic bending rather than independent as argued in Hill (1950). It should also be pointed out that the moment capacity of a strain-hardening material plate is significantly larger than that of its elastic–perfectly plastic material counterpart, as shown in the table. Therefore, the strain-hardening effect of material should be taken into account in an actual design to make more efficient use of the material.

6. CONCLUSIONS

1. The present finite deformation elasto-plastic analytical solution for a strain-hardening wide plate subjected to pure bending is derived by using the elastic linear-hardening

material model to include the strain-hardening effect of the material, the logarithmic (Hencky) strain to measure the large deformation of the plate, and a finite-strain version of Hencky's deformation theory to represent the constitutive relations of the problem.

2. Being based on the elastic linear-hardening material model, as idealized as it is, this solution has an additional use, viz. it may provide "benchmarks" for the validation of computer codes which are usually employed to numerically solve more general problems of this type. In such computer codes a known (arbitrary) stress–strain curve of an actual strain-hardening material is typically modelled by a piecewise linear representation in conjunction with a suitable iteration procedure (Owen and Hinton, 1980).

3. The finite-strain version of Hencky's deformation theory for an isotropic incompressible elasto-plastic (non-linear–elastic) solid is derived in tensor form directly from its small-strain counterpart by using the Hencky strain and the Cauchy stress, which has no restrictions on the yield criterion to be used. The von Mises yield criterion is adopted in this paper, which is generally regarded to be more accurate than the Tresca yield criterion, the other most commonly used yield criterion (Johnson and Mellor, 1983).

4. The present solution can be reduced to other specific solutions. All of the three specific solutions derived here are of practical value in engineering design. However, it should be pointed out that the specific solution for the plate in a fully yielded state presented in Section 5.2 is only an idealized one. From displacement continuity conditions, it follows that there always exists a small-strain elastic domain near the neutral surface in the pure bending process of an elastic–plastic plate no matter how large the bending moment couple may be (Johnson and Mellor, 1983). Hence, this specific solution may only well represent the case for which the size of the central elastic domain is negligibly small when compared with the sizes of the two plastic domains. Nevertheless, it does provide an upper-bound solution for the finite deformation pure bending of a wide plate of elastic linear-hardening material.

5. The present solution, as an exact one, is subject to the assumptions of material incompressibility, homogeneity and isotropy; no body forces acting; isotropic hardening; monotonic loading without unloading; and edge effects negligible. For other cases, e.g. unloading occurs after bending (Chen, 1962); bifurcations take place during bending (Triantafyllidis, 1980); and the plate material is compressible (Eason, 1960; Budiansky, 1958), further investigations may be required with regard to the applicability of this solution.

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